

# Partitioning twofold triple systems into complete arcs

K.T. Phelps\*

*Division of Mathematics, Department of Algebra, Combinatorics & Analysis, 120 Math Annex,  
Auburn University, AL 36849, USA*

M.J. de Resmini

*Dipartimento di Matematica, Università di Roma "La Sapienza", I-00185 Roma, Italy*

Received 25 July 1989

Revised 5 November 1990

## Abstract

Phelps, K.T. and M.J. de Resmini, Partitioning twofold triple systems into complete arcs, *Discrete Mathematics* 104 (1992) 273–280.

We establish that for all  $s$ , there exists a design with parameters  $(s^2, 3, 2)$  such that the points can be partitioned into  $s$  complete  $s$ -arcs. Furthermore we present a general technique which applies to the construction of designs  $(s^2, 4, 1)$  possessing a partition into  $s$  complete  $s$ -arcs. This gives a partial solution to this question as well.

## 1. Introduction

A block design with parameters  $(v, k, \lambda)$  is usually denoted by a pair  $(V, B)$  where  $V$  is a  $v$ -set and  $B$ , a collection of  $k$ -subsets of  $V$  containing each pair of  $V$   $\lambda$  times. As with hypergraphs, an independent set in a design  $(V, B)$  is a subset  $S \subseteq V$ , such that no block of  $B$  is contained in  $S$ . Assuming that  $|b| > 2$  for all  $b \in B$ , we can be more restrictive and define a set  $S \subseteq V$  to be an arc if  $|S \cap b| \leq 2$  for all  $b \in B$ . An arc  $S$  is said to be complete if  $S$  is maximal (with respect to set inclusion).

Arcs play an important role in finite projective planes where they have been extensively studied. Arcs in block designs have also been studied from several perspectives ([2, 4, 6, 7]). In this paper we consider the existence of designs (primarily  $(v, 3, 2)$ -designs) which have a certain extremal configuration involving arcs. In particular, we wish to construct designs in which the point set  $V$  can be partitioned into complete arcs of minimum size.

\* Researched supported by NSF grant DMS-8802261.

For a block design  $(v, 3, \lambda)$  to have a complete  $s$ -arc, we must have,

$$\lambda \binom{s}{2} + s \geq v$$

When  $\lambda = 2$ , this reduces to  $s^2 \geq v$ . Our question then is: does there exist a two-fold triple system of order  $s^2$  which can be partitioned into  $s$  complete  $s$ -arcs for all  $s \geq 2$ . For the sake of brevity we will call such a partition a complete  $s$ -coloring.

The same essential question for the case  $\lambda = 1$  (Steiner triple systems) was settled for the most part in a previous paper [5]. The techniques developed in this paper are different and can be generalized to block designs with parameters  $(v, 4, 1)$ .

We divide the problem into cases:  $s \equiv 0, 1$ , or  $2 \pmod{3}$ .

## 2. The cases $s \equiv 0, 1 \pmod{3}$

Two-fold triple systems, briefly TTS( $s$ ), exist for all  $s \equiv 0, 1 \pmod{3}$ . With the exception of  $s = 6$ , the spectrum for Mendelsohn triple system, briefly MTS( $s$ ), is the same. As is well known, a Mendelsohn triple system is equivalent to an idempotent quasigroup invariant under cyclic conjugation. When we refer to a MTS( $s$ ) having an orthogonal mate, we mean the latin square equivalent to the MTS( $s$ ) has an orthogonal mate.

First, two elementary but necessary facts.

**Lemma 2.1.** *There exists a latin square of order  $s = 3t$ ,  $s \neq 6$ , with  $t$  holes of size 3.*

**Lemma 2.2** [1]. *There exists a resolvable MTS( $s$ ) for all  $s \equiv 0 \pmod{3}$ ,  $s \neq 6$ .*

**Theorem 2.3.** *There exists a TTS( $s^2$ ) with a complete  $s$ -coloring for all  $s \equiv 0 \pmod{3}$ . (except possibly  $s = 6$ )*

**Proof.** Let  $V = Z_s \times Z_s$  and let  $(Z_s, B)$ ,  $(Z_s, B')$  denote resolvable MTS( $s$ ). Let  $T_i, T'_i$   $i = 1, 2, \dots, s-1$  denote the resolution classes in  $B$  and  $B'$  respectively. For each  $i$ , let  $L_i$  denote a latin square of order  $s$  with the holes of size 3 corresponding to the 3-subsets in  $T'_i$ . We consider  $L_i$  as a set of ordered triples of  $Z_s$ , defined in the standard manner.

For each  $i = 1, 2, \dots, s-1$ , and for each ordered triple (with fixed but arbitrary ordering)  $(a_0, a_1, a_2) \in T_i$  we include in  $B^*$  all triples:

- (1)  $\{(a_0, u), (a_1, v), (a_2, w)\} \in B^*$ , for each  $(u, v, w) \in L_i$
- (2) for each  $b \in T'_i$ ,  $b = (x_0, x_1, x_2)$

$$\{(a_r, x_j), (a_r, x_{j+1}), (a_{r+1}, x_{j+2})\} \in B^*,$$

for  $r = 0, 1, 2$  and for  $j = 0, 1, 2$  with all subscripts reduced mod 3.

- (3)  $\{(a_0, x), (a_1, x), (a_2, x)\}$  for all  $x \in Z_s$ .

Then it is straightforward to see that  $(V, B^*)$  as constructed above is a  $TTS(s^2)$ . Moreover, the sets  $\{a\} \times Z_s$  are clearly disjoint  $s$ -arcs. To establish that they are complete we need to show that the sets  $\{a\} \times Z_s \cup \{(b, x)\}$  each contain a triple.

First the ordered pair  $(a, b)$  is contained in exactly one triple, say  $(a, b, c)$ , of  $B$ . Assume  $(a, b, c) \in T_i$ , the  $i$ th resolution class. Next  $x$  is in a unique triple, say  $(x, y, z)$  of  $T_i'$  and hence there is a triple  $t = \{(a, y), (a, z), (b, x)\}$  in  $B^*$ . Our set contains (exactly) one triple for each choice of  $(b, x) \in \{a\} \times Z_s$  and thus  $\{a\} \times Z_s$  is complete.  $\square$

The above argument does not handle the case  $s = 6$ , which we will construct in the next section.

For the next case,  $s \equiv 1 \pmod{3}$ , we need  $MTS(s)$  with unipotent orthogonal mates.

**Lemma 2.4** [3]. *For all  $s \equiv 1 \pmod{3}$ ,  $s \neq 10$  or  $19$  there exists a pairwise balanced design with blocksize  $\{4, 7\}$  (and at most one block of size 7).*

**Corollary 2.5.** *For all  $s \equiv 1 \pmod{3}$ , there exists a latin square equivalent to an  $MTS(s)$  which has a unipotent orthogonal mate, except possibly  $s = 10, 19$ .*

**Proof.** There exists an  $MTS(4)$  with a unipotent orthogonal mate and an  $MTS(4)$  and  $MTS(7)$  with idempotent orthogonal mates. Using the design from Lemma 2.4 and assuming all blocks through 0 are of size 4, we put an  $MTS(4)$  and its unipotent orthogonal mate on these blocks (0 is the unipotent element, see Table 1). On all other blocks, place an  $MTS(4)$  or  $MTS(7)$  with an idempotent orthogonal mate less the diagonal elements.  $\square$

Table 1

MTS(4) with unipotent mate	MTS(4) with idempotent mate	MTS(7) with idempotent mate
0, 0, 0, 0	1 1 1 1	(0, 0, 0, 0)
1, 1, 1, 0	2 2 2 2	(1, 2, 4, 0)
2, 2, 2, 0	3 3 3 3	(2, 4, 1, 0)
3, 3, 3, 0	4 4 4 4	(4, 1, 2, 0)
0, 1, 2, 1	1 2 3 4	(6, 5, 3, 0)
1, 2, 0, 2	2 3 1 4	(5, 3, 6, 0)
2, 0, 1, 3	3 1 2 4	(3, 6, 5, 0)
0, 3, 1, 2	1 4 2 3	(vectors developed mod 7)
3, 1, 0, 3	4 2 1 3	
1, 0, 3, 1	2 1 4 3	
0, 2, 3, 3	1 2 4 2	
2, 3, 0, 1	3 4 1 2	
3, 0, 2, 2	4 1 3 2	
1, 3, 2, 3	2 4 3 1	
3, 2, 1, 1	4 3 2 1	
2, 1, 3, 2	3 2 4 1	

**Lemma 2.6.** *The complete digraph on  $Z_s$ ,  $s \equiv 1 \pmod{3}$  can be decomposed into isomorphic directed 2-factors where each 2-factor consists of 2 directed cycles of length 2 and  $(s-4)/3$  directed 3-cycles, except possibly  $s = 10, 19$ .*

**Proof.** Let  $(Z_s, D)$  denote the underlying design from Lemma 2.4. Take the MTS( $s$ ) and its unipotent orthogonal mate as constructed in Corollary 2.5. Assume this orthogonal array is defined on the set  $Z_s$ , and 0 is the unipotent element. For each  $i = 1, 2, \dots, s-1$  take all vectors  $(x_1, x_2, x_3, i)$  in the array and form  $F_i$  as follows:

- (1)  $\{x_1, x_2, x_3, i\} \subseteq b \in D$  and  $0 \in b$  then  $(x_1, x_2) \in F_i$ .
- (2) otherwise the directed 3-cycle  $(x_1, x_2, x_3) \in F_i$ .

The proof follows directly from the properties of the 3 orthogonal arrays used in this construction (see Table 1).  $\square$

We next construct a ‘partial’ triple system on  $Z_3 \times Z_s$  such that every pair  $\{(i, x), (j, y)\}$  with  $i \neq j$  is covered by exactly one triple and the pairs in  $\{i\} \times Z_s$  which are covered by a triple, form a 2-factor isomorphic to those of Lemma 2.6 (i.e., two 2-cycles and  $(s-4)/3$  3-cycles).

Let  $S$  denote the orthogonal array formed in Corollary 2.5 and let  $L_j = \{(x, y, z) \mid (x, y, z, w) \in S \text{ and } w \neq 0 \text{ or } j\}$  for a fixed  $j, j \neq 0$ . Let  $V = Z_3 \times Z_s$  and define  $D^*$  as follows:

- (1)  $\{(0, x), (1, y), (2, z)\}$ , for all  $(x, y, z) \in L_j$
- (2) For each other  $(x, y, z, j) \in S$  with  $\{x, y, z, j\} \subseteq b \in D$  and  $0 \in b$ ,
  - $\{(0, x), (0, y), (1, y)\}$ ,
  - $\{(1, y), (1, z), (2, z)\}$ ,
  - $\{(2, z), (2, x), (0, x)\}$ .
- (3) For each other  $(x, y, z, j) \in S$  (i.e.  $(x, y, z) \in F_j$ ),
  - $\{(i, x), (i, y), (i+1, y)\}$ ,
  - $\{(i, y), (i, z), (i+1, z)\}$ ,
  - $\{(i, z), (i, x), (i+1, x)\}$  for  $i = 0, 1, 2 \pmod{3}$ .

Note the blocks (2) come from the unique MTS(4) containing both 0 and  $j$ .

Clearly  $(V, D^*)$  defined above is a partial triple system as desired. We are now ready for our next theorem.

**Theorem 2.7.** *For  $s \equiv 1 \pmod{3}$  (except possibly  $s = 7, 10, 19$ ), there exists a TTS( $s^2$ ) with a complete  $s$ -coloring.*

**Proof.** Take an MTS( $s$ )  $(z_s, B)$  and a decomposition of the complete digraph on  $Z_s$  into 2-factors as described above, defined on each of the sets  $\{i\} \times Z_s$ ,  $i = 0, 1, \dots, s-1$ . Again let the point set be  $Z_s \times Z_s$ . For each triple  $(a, b, c) \in B$  select an unused 2-factor from the 2-factorizations defined on the sets  $\{a\} \times$

$Z_s$ ,  $\{b\} \times Z_s$  and  $\{c\} \times Z_s$  respectively. On the set  $\{a, b, c\} \times Z_s$  place an isomorphic copy of  $D^*$  (described above) so that the pairs in  $\{a\} \times Z_s$ ,  $\{b\} \times Z_s$ , and  $\{c\} \times Z_s$  covered by triples of  $D^*$ , correspond to the pairs in the selected 2-factors on those sets. Repeating this for each triple of  $B$  gives a  $\text{TTS}(s^2)$  with the sets  $\{i\} \times Z_s$ ,  $i \in Z_s$  as complete arcs. The remainder of the proof is similar to that of the previous theorem and is left to the reader.  $\square$

### 3. The case $s \equiv 2 \pmod{3}$

Before proceeding with the next construction we need to explain what a complete  $k$ -coloring of a latin square or orthogonal array of order  $k^2$  means. Let  $L$  be an idempotent latin square on the set  $V$ , (i.e.  $(x, x, x) \in L$ , for all  $x \in V$ ). A complete  $k$ -arc in  $L$  is a subset  $S \subseteq V$  such that whenever  $(x, y, z) \in L$  and  $x, y \in S$  then  $z \in V \setminus S$  unless  $x = y$  and moreover for each  $z \in V \setminus S$  there is at least one ordered pair  $(x, y)$ ,  $x, y \in S$  such that  $(x, y, z) \in L$ .

Naturally, a complete  $k$ -coloring of a latin square of order  $k^2$  is a partition of the set  $V$  into  $k$  complete  $k$ -arcs. There are several ways that this idea extends to orthogonal arrays (or equivalently sets of mutually orthogonal latin squares). The most restrictive definition says that an idempotent orthogonal array  $\text{OA}(p, k^2)$  with row vectors  $(x_1, x_2, \dots, x_p) \in L$  has a complete  $k$ -coloring if it is a complete  $k$ -coloring for the latin square  $L(i, j, l) = \{(x_i, x_j, x_l) \mid (x_1, x_2, \dots, x_p) \in L\}$  for each triple of distinct indices  $(i, j, l)$ . We actually only need a weaker version of this; namely, it must be true only for some choices of indices  $(i, j, k)$  but we will use the stronger definition.

Latin squares and orthogonal arrays can be constructed from block designs and under certain conditions will inherit the same properties of these designs.

**Lemma 3.1** (a). *If there is  $\text{MTS}(k^2)$  with a complete  $k$ -coloring then the corresponding latin square has a complete  $k$ -coloring.*

(b) *If there is a  $(k^2, 4, 1)$  design with a complete  $k$ -coloring then there is an orthogonal array  $\text{OA}(4, k^2)$  with a complete  $k$ -coloring.*

**Proof.** Statement (a) is obvious. Statement (b) is true if the idempotent  $\text{OA}(4, 4)$  has a complete 2-coloring which it does (see Table 1).  $\square$

The next theorem is designed with two designs in mind:  $(v, 3, 2)$  and  $(v, 4, 1)$ -designs.

**Theorem 3.2.** *If there is an  $(s^2, p, \lambda)$  design with a complete  $s$ -coloring, a  $(k^2, p, \lambda)$  design with a complete  $k$ -coloring and an  $\text{OA}(p, k^2)$  with a complete  $k$ -coloring then there exists an  $(k^2 s^2, p, \lambda)$  design with a complete  $ks$ -coloring.*

**Proof.** Let  $K_1, K_2, \dots, K_k$  be the complete  $k$ -coloring of the  $(k^2, p, \lambda)$  design and the  $\text{OA}(p, k^2)$  on the set  $K = K_1 \cup K_2 \cup \dots \cup K_k$ . Similarly let  $S_1, \dots, S_s$  be the complete  $s$ -coloring of the  $(s^2, p, \lambda)$  design on the set  $S = S_1 \cup \dots \cup S_s$ . Denote the two designs by  $(K, B(K))$  and  $(S, B(S))$  respectively and the orthogonal array by  $(K, L(K))$ . We construct a design on  $S \times K$  in a fairly standard manner; conceptually, we are ‘blowing up’ the points of  $S$ , replacing each by a copy of the set  $K$ . Formally stated we have:

(1) for each  $a \in S$  include in  $D$ , the blocks of the  $(k^2, p, \lambda)$  design defined on the set  $\{a\} \times K$  with complete  $k$ -arcs  $\{a\} \times K_i$

(2) for each block  $b \in B(S)$ ,  $b = \{a_1, a_2, \dots, a_p\}$  and each  $(x_1, x_2, \dots, x_p) \in L(K)$  include the block  $\{(a_1, x_1), (a_2, x_2), \dots, (a_p, x_p)\}$ .

Clearly  $(S \times K, D)$  is a  $(k^2 s^2, p, \lambda)$ -design with arcs  $S_i \times K_j$ . To see that these arcs are complete, consider any point  $(a, x)$  not in  $S_i \times K_j$ ; then either:

(1)  $a \in S_i$ ,  $x \notin K_j$  but  $\{a\} \times K_j$  is a complete hence  $\{(a, x)\} \cup \{a\} \times K_j$  contains a triple

(2)  $a \notin S_i$ . Since  $S_i$  is complete there is a block  $l \in B(S)$  such that  $l \cap (S_i \cup \{a\}) = \{a, b, c\}$ , with  $b, c \in S_i$ .

If  $x \in K_j$ , then there is a block  $l \in D$  containing  $(b, x)$ ,  $(c, x)$ , and  $(a, x)$  since  $L(K)$  is idempotent.

If  $x \in K_i$  then the fact that  $L(K)$  has a complete coloring guarantees that  $L(b, c, a)$  has this complete  $k$ -coloring which in turn means that there exists  $y, z \in K_i$  such that  $(y, z, x) \in L(b, c, a)$  and thus a block  $l^*$  containing  $(b, y)$ ,  $(c, z)$  and  $(a, x)$ .  $\square$

Note that in our construction we used an arbitrary ordering of the elements of the blocks  $b \in B(S)$ . Had we bothered to order the elements of  $b$ , based on the intersections with the arcs  $S_i$  (i.e., when  $|b \cap S_i| = 2$ ) then for the argument in the above paragraph we would only need that  $L(K)$  had the ‘weaker’ complete  $k$ -coloring conditions mentioned before.

Now we consider some corollaries relative to  $\text{TTS}(v)$ .

**Corollary 3.3.** *There exist  $\text{TTS}(s^2)$  with complete  $s$ -coloring for  $s = 6, 10$ .*

**Proof.** There exists an  $(25, 4, 1)$  design with a complete 5-coloring and hence a  $\text{MTS}(25)$  with a complete 5-coloring (Theorem 3.8). Applying the theorem with  $k = 2$  and  $s = 5$  gives use an  $\text{MTS}(10^2)$ . Similarly using  $k = 2, s = 3$  the above theorem gives  $\text{TTS}(6^2)$  with a complete 6-coloring.  $\square$

**Corollary 3.4.** *There exists a  $\text{TTS}(s^2)$  with a complete  $s$ -coloring for all  $s \equiv 2 \pmod{6}$  (except possibly  $s = 2 \cdot 7$  and  $s = 2 \cdot 19$ ).*

**Proof.** Since  $s = 6t + 2 = 2(3t + 1)$  we simply apply Theorem 3.2 where  $k = 2$  and  $3t + 1$ . The only possible exceptions are  $2 \cdot 7$  and  $2 \cdot 19$ , but these cases will be handled shortly.  $\square$

The last remaining case is when  $s \equiv 5 \pmod{6}$ . Here all prime factors of  $s$  are congruent to 1 or 5 mod 6 so we can write  $6t + 5 = ks'$  where  $k \equiv 5 \pmod{6}$  is a prime and  $s' \equiv 1 \pmod{6}$ .

**Theorem 3.5.** *There exists a  $\text{MTS}(s^2)$  for all  $s \equiv 1 \pmod{6}$ , with a complete  $s$ -coloring.*

**Proof.** For all  $s \equiv 1 \pmod{6}$  there exists a cyclic Steiner triple system  $(Z_s, B)$ . For each base block  $\{a, b, c\}$  form 2 ordered triples  $(a, b, c)$  and  $(-a, -b, -c) \pmod{s}$ . The differences are  $(x, y, z)$  and  $(-x, -y, -z)$  respectively (i.e.  $b - a \equiv x \pmod{s}$ ,  $c - b \equiv y \pmod{s}$ ,  $a - c \equiv z \pmod{s}$ , etc.) Let these be the base triples for a cyclic latin square of order  $s$ . Each pair of base triple  $(a, b, c)$ ,  $(-a, -b, -c)$  in the  $\text{MTS}(s)$  are also base triple in the Latin square. Delete the whole orbit of these triples from the latin square and let  $L(a, b, c)$  denote the remaining triples.

For each base triple  $(a, b, c)$  in  $\text{MTS}(s)$  with difference triple  $(x, y, z)$  construct the following base triples:

- (1)  $((a, 0), (a, 2x), (b, x)), ((b, 0), (b, 2y), (c, y)), ((c, 0), (c, 2z), (c, z))$ .
- (2)  $((a, u), (b, v), (c, w))$  for each base triple  $(u, v, w) \in L(a, b, c)$ .

Letting  $Z_s \times Z_s$  act on these triples in the natural way gives a  $\text{MTS}(s^2)$  with complete  $s$ -arcs  $\{a\} \times Z_s$ ,  $a \in Z_s$ .

This construction is similar to one utilized in a previous paper [5].  $\square$

**Corollary 3.6.** *There exist  $\text{MTS}(s^2)$ , for  $s = 7, 19, 2 \cdot 7, 2 \cdot 19$  which have a complete  $s$ -coloring.*

**Corollary 3.7.** *If there exists  $\text{MTS}(k^2)$  with a complete  $k$ -coloring for each prime  $k \equiv 5 \pmod{6}$  then there exists a  $\text{TTS}(s^2)$  for all  $s \equiv 5 \pmod{6}$ .*

All that remains to be done is to construct  $\text{TTS}(s^2)$  with complete  $s$ -coloring for all primes,  $s \equiv 5 \pmod{6}$ .

The following construction of an  $\text{MTS}(p^2)$   $p \equiv 5 \pmod{6}$ , is essentially the well known construction of Steiner triple systems from finite fields. Let  $V = Z_p \times Z_p$  and let  $\alpha \in V$  denote the primitive element in the finite field of order  $p^2$ . Let

$$p = 6k + 5 \quad \text{and} \quad p^2 - 1 = (p + 1)(p - 1) = 6t.$$

The base triples (orbit representatives) include:

$$(\alpha^i, \alpha^{2t+i}, \alpha^{4t+i}), \quad \text{and} \quad (-\alpha^i, -\alpha^{2t+i}, -\alpha^{4t+i}) \quad \text{for } i = 0, 1, \dots, t-1.$$

The abelian  $p$ -group  $Z_p \times Z_p$  acts on these elements in the natural way. The result is a transitive  $\text{MTS}(p^2)$ . Let  $(Z_p \times Z_p, B)$  denote this  $\text{MTS}(p^2)$ .

**Theorem 3.8.** *The  $\text{MTS}(p^2)$ ,  $(Z_p \times Z_p, B)$  constructed above, has a complete  $p$ -coloring.*

**Proof.** We claim that the ground field  $F = \text{GF}(p) \subseteq \text{GF}(p^2)$  and its translates  $F + a^j\alpha$ ,  $a^j \in F^*$  are complete  $p$ -arcs. It suffices to consider only  $F$ . Note  $F^* = \{\alpha^{j(p+1)} \mid j = 0, 1, \dots, p-2\}$ . Consider the differences in an arbitrary base block  $\pm(\alpha^i, \alpha^{2t+i}, \alpha^{4t+i})$ ; they are  $\pm\alpha^i(\alpha^{2t} - 1)$  ( $1, \alpha^{2t}, \alpha^{4t}$ ). For a block  $b \in B$ , to be contained in the subfield  $F$ , all 3 differences must be contained in that subfield. So we may assume that,

$$\pm\alpha^i(\alpha^{2t} - 1) = \alpha^{j(p+1)} \in F$$

But then the difference  $\alpha^{2t+j(p+1)}$  is not in  $F$  because  $p+1$  does not divide  $2t$  (since  $6t = (p-1)(p+1)$  and  $p \equiv 5 \pmod{6}$ ). To see that each arc is complete we again only consider base blocks which have a difference in  $F$ . Hence we can again assume that  $\pm\alpha^i(\alpha^{2t} - 1) = \alpha^{j(p+1)}$  and thus one block in the orbit will be  $(0, \alpha^{j(p+1)}, \alpha^{2t+j(p+1)} + \alpha^{j(p+1)})$  for each  $j = 0, 1, \dots, p-2$ . Since,  $\alpha^{2t+j(p+1)} + \alpha^{j(p+1)}$  is not in  $F$ , we know  $(\alpha^{2t} + 1)\alpha^{j(p+1)} \in F + a\alpha$  and thus for each  $j = 0, 1, \dots, p-2$ ,  $(\alpha^{2t} + 1)\alpha^{j(p+1)}$  is a representative of a different translate. Thus  $F$  is a complete  $p$ -arc, and  $B$  has a complete  $p$ -coloring.  $\square$

#### 4. Conclusion

We point out that Theorem 3.2 applies to designs  $(v, 4, 1)$  as well. Since there exist  $(s^2, 4, 1)$  designs with complete  $s$ -coloring for  $s = 2, 5$  and  $7$ , this theorem gives an infinite class of such designs. Given this theorem, a complete solution of this problem for  $(v, 4, 1)$ -designs is quite conceivable. (See de Resmini [8], for a complete discussion of this problem.)

#### References

- [1] J.C. Bermond, A. Germa and D. Sotteau, Resolvable decompositions of  $K_n^*$ , J. Combin. Theory Ser. A 26 (1979) 179–185.
- [2] M. de Brandes and V. Rodl, Steiner triple systems with small maximal independent sets, Ars Combin. 17 (1984) 15–19.
- [3] A.E. Brouwer, Optimal packings of  $K_4$  into  $K_n$ , J. Combin. Theory Ser. A 26 (1979) 278–297.
- [4] C.J. Colbourn, J. Dinitz and D. Stinson, Spanning sets and scattering sets in Steiner triple systems, J. Combin. Theory Ser. A 57 (1991) 46–59.
- [5] C.J. Colbourn, K.T. Phelps, A. Rosa and M.J. de Resmini, Partitioning Steiner triple systems into complete arcs, Discrete Math. 89 (1991) 149–160.
- [6] K.T. Phelps and V. Rodl, Steiner triple systems with minimum independence number, Ars Combin. 21 (1986) 167–172.
- [7] M.J. de Resmini, On sets of type  $(m, n)$  in Steiner systems  $S(2, l, v)$ , in: Finite geometries and Designs (London Math. Society, London, 1981) 104–113.
- [8] M.J. de Resmini, On complete arcs in Steiner systems  $S(2, 3, v)$  and  $S(2, 4, v)$ , Discrete Math. 77 (1989) 65–73.